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Analytic properties of gas hard core spheres in a continuous system of statistical classical mechanics

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Abstract
We find the resolvent at the Kirkwood–Salsburg integral operator in a finite volume for hard core potentials. We also construct the resolvent at the Kirkwood–Salsburg operator in the thermodynamical limit for values of chemical activity $z > e^{C(\beta)d^{-1}}$ and for non-negative potentials. Therefore the phase transition does not occur for these values of chemical activity.

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1. Introduction

We describe the states in statistical mechanics by the correlation function or equivalently by the Gibbs measures [1, 2]. In the grand canonical ensemble, the phase transition can occur only for values of chemical activity z such that the correlation functions are not analytic. The main tool to study the analytic properties of the correlation function in the thermodynamical limit is the Kirkwood–Salsburg (KS) integral equations [3]. We can write the KS equation in the operator form (16) [3]. The correlation functions are equal to the resolvent of the KS operator acting on the vector I (15); so the problem of phase transition in gases can be solved by searching the behaviour of the spectral set in the thermodynamical limit [4, 5]. In a standard approach the only existing results on the spectrum of the KS operator is the estimation of the spectral radius by means of its norm in the Banach space. This approach gives us the analyticity of the correlation function for low values of density and equivalently chemical activity [1, 3, 6]. In this paper we consider the Kirkwood–Salsburg equation on the cyclic space of the KS operator. This is the smallest possible Banach space which contains all the correlation functions [5]. In the first section exploring properties of this cyclic space we find the resolvent of the KS operator for the potential with a hard core for the finite volume case. It is an important new result which we can utilize in many ways. In the next section we show one possible application of this result. We introduce the operator (53) for the non-negative regular potentials in the infinite volume space. This operator has a similar construction to the resolvent of the KS operator in a finite volume. We prove that for values of chemical activity $z > e^{C(\beta)d^{-1}}$ the operator is in fact the resolvent of the KS operator in the thermodynamical limit. This allows us to conclude that the thermodynamical limit of the correlation functions is an analytic function of positive chemical activity larger than $e^{C(\beta)d^{-1}}$; thus the phase transition does not occur for such values of chemical activity. It is worthwhile to stress that the region of analyticity obtained is the only one known for gases with large values of density. We conclude that the phase transition is possible only for values of chemical activity $z \in [\{(e-1)C(\beta)\}^{-1}, e^{C(\beta)d^{-1}}]$ [5, 10].

2. Notation and definition

Let $x_i \in R^\nu$ for $i = 1, \ldots, n$, $(x)_n = (x_1, \ldots, x_n)$ and $\Phi : R^+ \to R$ be a Lebesgue measurable function. We assume that there does exist such $r > 0$ that for each $a \in [0, r]\Phi(a) = \infty$ and Φ is bounded below. Under these conditions, $\Phi(|x_i - x_j|)$ stands for the two-body interaction potential for i and j particles, both with a hard core. The potential energy is given by

$$U(x)_n = \sum_{1\leqslant i<j\leqslant n} \Phi(|x_i - x_j|) \tag{1}$$

and

$$W((x)_p, (y)_l) = \sum_{1\leqslant i\leqslant p, 1\leqslant j\leqslant l} \Phi(|x_i - y_j|) \tag{2}$$

$$U((x)_p, (y)_l) = U(x)_p + W((x)_p, (y)_l) + U(y)_l. \tag{3}$$

Let Λ be a bounded region in R^ν, χ_Λ the characteristic function of the Λ and

$$\chi_\Lambda(x)_n = \prod_{i=1}^{n} \chi_\Lambda(x_i). \tag{4}$$

Let us define the following sequences of functions:

$$\hat{\rho}_\Lambda\{a_l\}_{l=1}^{L}(x)_p = \sum_{n=0}^{L-p} \frac{a_{n+p}}{n!} \int_{\Lambda^n} e^{-\beta U((x)_p,(y)_n)}\, d(y)_n \tag{5}$$

$a_l \in R, l = 1, \ldots, L$, where L is the greatest possible number of particles contained in the region Λ . In this section we consider the interaction with a hard core, so the number L is finite. Using (5) we write the correlation functions as follows:

$$\hat{\rho}_\Lambda \left\{ \frac{z^l}{\Xi_\Lambda(z)} \right\}_{l=1}^{L} (x)_p, \tag{6}$$

where

$$\Xi_\Lambda(z) = \sum_{l=0}^{L} \frac{z^l}{l!} \int_{\Lambda^l} e^{-\beta U(x)_l}\, d(y)_l \tag{7}$$

means the grand canonical partition function.

Let A_Λ and B_Λ denote two linear operators acting on sequences $\hat{\phi} = \{\phi(x)_n\}_{n=1}^{\infty}$ of the symmetric function $\phi(x)_n$ in the following way:

$$(A_\Lambda\phi)(x)_p = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \phi((x)_p, (y)_m)\, d(y)_m \tag{8}$$

$$(B_\Lambda\phi)(x)_p = \sum_{m=0}^{\infty}\frac{(-1)}{m!}\int_{\Lambda^m}\phi((x)_p,(y)_m)\,\mathrm{d}(y)_m. \tag{9}$$

Then we have

$$A_\Lambda B_\Lambda = B_\Lambda A_\Lambda = I \tag{10}$$

$$B_\Lambda\left(\hat{\rho}_\Lambda\{a_l\}_{l=1}^{L}\right) = \{a_l\,\mathrm{e}^{-\beta U(x)_l}\}_{l=1}^{L} \tag{11}$$

in the particular acting on the correlation functions. We obtain

$$B_\Lambda\left(\hat{\rho}\left\{\frac{z^l}{\Xi_\Lambda(z)}\right\}_{l=1}^{L}\right) = \left\{\frac{z^l}{\Xi_\Lambda(z)}\,\mathrm{e}^{-\beta U(x)_l}\right\}_{l=1}^{L}. \tag{12}$$

Using the operator notation we can write the KS equations on the space of the sequences $\hat{\varphi} = \{\varphi(x)_n\}_{n=1}^{\infty}$, where $\varphi(x)_n$ are the measurable complex-valued functions on R^n as follows:

$$\hat{\rho}_\Lambda - zK_\Lambda\hat{\rho}_\Lambda = I, \tag{13}$$

where

$$(K_\Lambda\hat{\rho}_\Lambda)(x_1) = \sum_{m=1}\frac{1}{m!}\int_{\Lambda^m}K(x_1,(y)_m)\rho(y)_m\,\mathrm{d}(y)_m \tag{14}$$

$$(K_\Lambda\hat{\rho}_\Lambda)(x)_n = \mathrm{e}^{-\beta W(x_1,(x)'_n)}(\rho(x)'_n) + \sum_{m=1}\frac{1}{m!}\int_{\Lambda^m}K(x_1,(y)_m)\rho((x)'_n,(y)_m)\,\mathrm{d}(y)_m$$

$$(x)'_n = (x_2,\ldots,x_n) \tag{15}$$

$$K(x_1,(y)_m) = \prod_{j=1}^{m}(\mathrm{e}^{-\beta\Phi(x_1-y_j)}-1), \qquad I = (\chi_\Lambda(x),0,\ldots,0).$$

3. Resolvent of the KS operator in a finite volume

In the case of finite volume the correlation functions (6) are the solution of the KS equation. The uniqueness of the solution of the KS equation depends on the spectral properties of the KS operator. This problem was considered by many authors (see [1] and references therein). Here we present another approach. We consider the KS operator on his cyclic subspace which allows us to easily get uniqueness of the solution of the KS equation. In the case of hard core potential the cyclic subspace is the finite dimensional. To follow this very important observation and using the operators A_Λ (8), B_Λ (9), we can write the KS operator as a matrix, and then we can calculate the resolvent of this matrix (23). It allows us to find the resolvent of the KS operator (38). At first we introduce the following cyclic space.

Definition 1. *Let $D(\lambda)$ be the L-dimensional linear space generated by the vectors $\left\{K_\Lambda^{l-1}I\right\}_{l=1}^{L}$, where $I = (\chi_\Lambda(x),0,\ldots,0)$.*

Proposition 1. *The space $D(\lambda)$ consists of the sequences $\hat{\rho}_\Lambda\{a_l\}_{l=1}^{L}$ (5).*

Proof. The vectors $K_\Lambda^{l-1}I$ are linear independent and they have the form (5) for $l = 1, 2, \ldots, L$. □

Definition 2. *[4] Let k_Λ be a linear operator on the space of the sequences $\{a_l\, \mathrm{e}^{-\beta U(x)_l}\}_{l=1}^{L}$ defined in following way:*

$$k_\Lambda\{a_l\, \mathrm{e}^{-\beta U(x)_l}\}_{l=1}^{L} = \{a_{l-1}\, \mathrm{e}^{-\beta u(x)_l}\}_{l=1}^{L}, \tag{16}$$

where $a_0 = -\sum_{l=1}^{L} \frac{a_1}{1!} \int_{\Lambda^l} \mathrm{e}^{-\beta U(x)_l} d(x)_l$.

With the use of definition 1 and equations (8), (9), it is easy to verify that the operator $K_\Lambda = A_\Lambda k_\Lambda B_\Lambda$ is the KS operator. Thus,

$$\begin{aligned}\left(K_\Lambda \hat{\rho}_\Lambda \{a_l\}_{l=1}^{L}\right)(x_1) &= \left(A_\Lambda k_\Lambda B_\Lambda \hat{\rho}_\Lambda \{a_l\}_{l=1}^{L}\right)(x_1)\\ &= \sum_{m=1} \frac{1}{m!} \int_{\Lambda^m} K(x_1, (y)_m) \rho\{a_l\}_{l=1}^{L} (y)_m \,\mathrm{d}(y)_m \end{aligned}\tag{17}$$

$$\begin{aligned}\left(K_\Lambda \hat{\rho}_\Lambda \{a_l\}_{l=1}^{L}\right)(x)_n &= \left(A_\Lambda k_\Lambda B_\Lambda \hat{\rho}_\Lambda \{a_l\}_{l=1}^{L}\right)(x)_n\\ &= \mathrm{e}^{-\beta W(x_1,(x)'_n)} \left(\rho_\Lambda \{a_l\}_{l=1}^{L} (x)'_n\right)\\ &\quad + \sum_{m=1} \frac{1}{m!} \int_{\Lambda^m} K(x_1, (y)_m) \rho\{a_l\}_{l=1}^{L} ((x)'_n, (y)_m) \,\mathrm{d}(y)_m. \end{aligned}\tag{18}$$

We can treat the functions $\mathrm{e}^{-\beta U(x)_l}$ as the basis and write the operator (16) in the form of the $L \times L$ matrix

$$\begin{pmatrix} -V & -\frac{1}{2!}\int_{\Lambda^2} \mathrm{e}^{-\beta U(x)_2} (x)_2 & \dots & \dots & -\frac{1}{L!}\int_{\Lambda^L} \mathrm{e}^{-\beta U(x)_L} (x)_L \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{19}$$

and equivalently

$$(k_\Lambda)_{kl} = \begin{cases} -\frac{1}{l!}\int_\Lambda \mathrm{e}^{-\beta U(x)_l} \,\mathrm{d}(x)_l & \text{for } k = 1 \\ 1 & \text{for } L \geqslant k \geqslant 2\, \mathrm{i} l = k-1 \\ 0 & \text{other.} \end{cases} \tag{20}$$

We can calculate the eigenvalues polynomial of this matrix. It is equal to $Q_\Lambda(\lambda) = z^{-L}\,\Xi_\Lambda(z)$, where $\lambda = z^{-1}$.

We define

$$Q^k(\lambda) = \sum_{l=k+1}^{L} \frac{\lambda^{L-l}}{l!} \int_{\Lambda^l} \mathrm{e}^{-\beta U(x)_l} \,\mathrm{d}(y)_l \tag{21}$$

$$Q_k(\lambda) = \sum_{l=0}^{k} \frac{\lambda^{L-l}}{l!} \int_{\Lambda^l} \mathrm{e}^{-\beta U(x)_l} \,\mathrm{d}(y)_l. \tag{22}$$

Now, we can write the matrix inverse to $(\lambda I - k_\Lambda)$ as

$$(\lambda I - k_\Lambda)^{-1}_{kl} = \begin{cases} -\frac{1}{\lambda^{k-l+1}} \frac{Q_{l-1}(\lambda)}{Q_\Lambda(\lambda)} & k \leqslant l \\ \frac{1}{\lambda^{k-l+1}} \frac{Q^{l-1}(\lambda)}{Q_\Lambda(\lambda)} & k > l \end{cases} \tag{23}$$

where $k, l = 1, 2, \dots, L$ and $Q_\Lambda(\lambda) = Q_k(\lambda) + Q^k(\lambda)$.

Our next aim is to find the resolvent of the operator K_Λ on the space $D(\lambda)$. We introduce the following pair of $L \times L$ additional matrices

$$E_\lambda = \begin{pmatrix} 1 & -\lambda & 0 & \dots & 0 \\ 0 & 1 & -\lambda & 0 & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & -\lambda \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \tag{24}$$

$$E_\lambda^{-1} = \begin{pmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^{l-1} \\ 0 & 1 & \lambda & \dots & \lambda^{l-2} \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \lambda \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \tag{25}$$

which allows us to find the resolvent of the KS operator. After an easy calculation we see that the matrix

$$\left(E_\lambda(\lambda I - k_\Lambda)E_\lambda^{-1}\right)_{kl} = \begin{cases} \lambda^{l-k-1} & k < l \\ 0 & l \leqslant k < L \\ \frac{Q_0(\lambda)+\dots+Q_{l-1}(\lambda)}{\lambda^{L-l+1}Q_\Lambda(\lambda)} & k = L \end{cases} \tag{26}$$

is much simpler than (21). The resolvent of K_Λ on the $D(\Lambda)$ can be written in the following way:

$$(\lambda I - K_\Lambda)^{-1} = A_\Lambda E_\lambda^{-1} B_\Lambda A_\Lambda E_\lambda(\lambda I - k_\Lambda)E_\lambda^{-1} B_\Lambda A_\Lambda E_\lambda B_\Lambda. \tag{27}$$

Our next consideration ensues the following definition.

Definition 3. *Let $N_\Lambda : D(\Lambda) \to D(\Lambda)$ denote the operator*

$$N_\Lambda = A_\Lambda \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} B_\Lambda. \tag{28}$$

One can observe that $N_\Lambda^L = 0$ and for $a_L = 0$

$$N_\Lambda K_\Lambda \rho_\Lambda \{a_l\}_{l=1}^L = \hat{\rho}_\Lambda \{a_l\}_{l=1}^L. \tag{29}$$

We can write (26) as the following sum:

$$\left(E_\lambda(\lambda I - k_\Lambda)E_\lambda^{-1}\right)_{kl} = \begin{cases} \lambda^{l-k-1} & k < l \\ 0 & k \geqslant l \end{cases} + \begin{cases} 0 & k \leqslant L \\ \frac{Q_0(\lambda)+\dots+Q_{l-1}(\lambda)}{\lambda^{L-l+1}Q_\Lambda(\lambda)} & k = L. \end{cases} \tag{30}$$

We can find the first expression of the sum (30) in the space $D(\Lambda)$

$$A_\Lambda \begin{cases} \lambda^{l-k-1} & k < l \\ 0 & k \geqslant l \end{cases} B_\Lambda = N_\Lambda + \lambda N_\lambda + \dots + \lambda^{L-2} N_\Lambda^{L-1}. \tag{31}$$

We need the following definition to write the second expression of the sum (30) in the space $D(\Lambda)$.

Definition 4. *Let us define the following projector:*

$$\prod\{\varphi(x)_n\}_{n=1}^{\infty} = \varphi(x)_1. \tag{32}$$

The operator $(K_\Lambda N_\Lambda - I)$ acting on $\hat{\rho}\{a_l\}_{l=1}^{L}$ gives $\hat{\rho}_\Lambda\{b_l\} = (b_1, 0, \ldots, 0)$; thus

$$\prod(K_\Lambda N_\Lambda - I) : D(\Lambda) \to R. \tag{33}$$

We get the following equality:

$$\begin{aligned}\prod&(K_\Lambda N_\Lambda - I)\left(I + \cdots + \lambda^{L-1}N_\Lambda^{L-1}\right)^2 \hat{\rho}_\Lambda\{a_l\}_{l=1}^{L}\\ &= a_1\frac{Q_0}{\lambda^L} + a_2\frac{Q_0+Q_1}{\lambda^{L-1}} + \cdots + a_L\frac{Q_0+\cdots+Q_{L-1}}{\lambda}.\end{aligned} \tag{34}$$

Exploring definition 3 and equation (28) we obtain

$$\begin{aligned}A_\Lambda E_\lambda(\lambda I - k_\Lambda)^{-1}E_\lambda^{-1}B_\Lambda &= \left(N_\Lambda + \lambda N_\lambda + \cdots + \lambda^{L-2}N_\Lambda^{L-1}\right)\\ &\quad - \left\{\frac{1}{(L-p)!}\int_{\Lambda^{L-p}} \mathrm{e}^{-\beta U((x)_p,(y)_{L-p})}\,\mathrm{d}(y)_{L-p}\right\}_{p=1}^{L}\\ &\quad \times \frac{1}{Q_\Lambda(\lambda)}\prod\left[(K_\Lambda N_\Lambda - I)\left(I + \lambda N_\Lambda + \cdots + \lambda^{L-1}N_\Lambda^{L-1}\right)^2\right].\end{aligned} \tag{35}$$

According to equation (25), we shall act on equation (30) with the operator $A_\Lambda E_\lambda^{-1}B_\Lambda$ on the left-hand side and with the operator $A_\Lambda E_\lambda B_\Lambda$ on the right-hand side. We notice that

$$A_\Lambda E_\lambda B_\Lambda = I - \lambda N_\lambda \tag{36}$$

$$A_\Lambda E_\lambda^{-1}B_\Lambda = I + \lambda N_\lambda + \cdots + \lambda^{L-1}N_\lambda^{L-1} \tag{37}$$

and as a result we get

$$\begin{aligned}(\lambda I - K_\Lambda)^{-1} &= -\left(N_\Lambda + \lambda N_\lambda + \cdots + \lambda^{L-2}N_\Lambda^{L-1}\right)\\ &\quad - \hat{\rho}_\Lambda\left\{\frac{\lambda^{L-l}}{Q_\Lambda(\lambda)}\right\}_{l=1}^{L}\prod\left[(K_\Lambda N_\Lambda - I)\left(I + \lambda N_\Lambda + \cdots + \lambda^{L-1}N_\Lambda^{L-1}\right)\right].\end{aligned} \tag{38}$$

Remark 1. We can check directly that the right-hand side of (33) is really the resolvent of K_Λ acting on $(\lambda I - K_\Lambda)$.

4. Resolvent of the KS operator in an infinite volume

In this section we find the resolvent of the KS operator in the thermodynamical limit for large values of the chemical activity z. Let us assume that the hard core two bodies potential $\Phi(|x_i - x_j|)$ is non-negative and regular,

$$C(\beta) = \int_0^\infty |\,\mathrm{e}^{-\beta\Phi(x)} - 1|\,\mathrm{d}x < \infty. \tag{39}$$

Definition 5. *Let E denote the Banach space of the sequences $\hat{\varphi} = \{\varphi(x)_n\}_{n=1}^{\infty}$, where $\varphi(x)_n$ are the measurable complex-valued functions on $R^{n\nu}$ and $d > 0$ with the norm*

$$\|\hat{\varphi}\| = \sup_{n\geqslant 1} d^{n-1} ess \sup_{(x)_n\in R^n} |\varphi(x)_n|. \tag{40}$$

Definition 6. *Let D be the subspace of E generated by the vectors $\{K^{l-1}I\}_{l=1}^{\infty}$, where $I = (1, 0, \ldots, 0)$.*

It is clear that the KS operator K is bounded on the spaces E

$$\|K\| \leqslant \mathrm{e}^{C(\beta)d^{-1}}. \tag{41}$$

We intend to find the resolvent of the KS operator studying the resolvent in the finite volume (38). At first we find another form of the operator N_Λ (26), more suitable for next consideration.

Let us define the two following operators:

$$(K_0(\Lambda)\hat{\varphi})\,(x)_p = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} K(x_1(y)_n)\varphi((x)_p(y)_n)\,\mathrm{d}(y)_n \tag{42}$$

$$\left(K_0^{-1}(\Lambda)\hat{\varphi}\right)(x)_p = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} K(x_1(y)_n)\varphi((x)_p(y)_n)\,\mathrm{d}(y)_n. \tag{43}$$

We have

$$K_0^{-1}(\Lambda)K_0(\Lambda) = K_0(\Lambda)K_0^{-1}(\Lambda) = I; \tag{44}$$

when $\Lambda = R^\nu$ we denote $K_0(R^\nu) = K_0$ and $K_0^{-1}(R^\nu) = K_0^{-1}$.

We can write each element $\varphi \in D$ in the following way:

$$\varphi = K\psi + \alpha I, \tag{45}$$

where $\psi \in D$ and α is a complex number. We see that

$$K_0^{-1}\varphi = \{\mathrm{e}^{-\beta W(x_i(x)'_p)}\psi(x)'_p\}_{p=2}^{\infty} + \alpha I. \tag{46}$$

Let us define the operator S as follows:

$$S\{\mathrm{e}^{-\beta W(x_i(x)'_p)}\varphi(x)'_p\}_{p=2}^{\infty} = \{\varphi(x)_p\}_{p=1}^{\infty}, \tag{47}$$

$$S(I) = 0. \tag{48}$$

For the finite range potentials we have the following equality,

$$\sup_{(x)_p} |\mathrm{e}^{-\beta W(x_i(x)'_p)}\varphi(x)'_p| = \sup_{(x)'_p} |\varphi(x)'_p|, \tag{49}$$

because for any configuration of the points $x_2, \ldots, x_p \in R^{p\nu}$ there is a point $x_1 \in R^\nu$ such that $\mathrm{e}^{-\beta W(x_i(x)'_p)} = 1$. The same result we get for the potentials tending to zero in infinity.

Using our notation we have in a finite volume

$$N_\Lambda\hat{\rho}_\Lambda\{a_l\}_{l=1}^{L} = S\{\varphi(x)'_p\}_{p=2}^{L} = SK_0^{-1}(\Lambda)\hat{\rho}_\Lambda\{a_l\}_{l=1}^{L}. \tag{50}$$

Therefore in the thermodynamical limit we define

$$N = SK_0^{-1} \tag{51}$$

and by (38), (44) we have the following estimation:

$$\|N\| \leqslant \mathrm{e}^{C(\beta)d^{-1}}. \tag{52}$$

We show that the resolvent of the KS operator in the infinite volume for small λ is as follows:

$$(\lambda I - K)^{-1} = -(N + \lambda N^2 + \cdots) - \hat{\rho}\prod[(KN - I)(I + \lambda N + \cdots)] \tag{53}$$

where $\hat{\rho}$ fulfils the KS equation (16).

Assuming that the right-hand side of (48) is the bounded operator, we can easily check that it is really the resolvent of K on the linear combination of the vectors $\left\{K^{l-1}I\right\}_{l=1}^{\infty}$. That set is dense in the space D, so we can extended the resolvent on all space D.

Our next step is to find an estimation of the correlation functions $\hat{\rho}$ in the thermodynamical limit for $z > 0$. It is a widely known result that exploring the Banach–Alaoglu theorem we get that $\hat{\rho} = \lim_{\Lambda\to\infty}\hat{\rho}_\Lambda$ for the real, positive z [3, 8] where

$$\hat{\rho}_\Lambda = \left\{ \frac{\sum_{n=0}^{L-p} \frac{z^{p+n}}{n!} \int_{\Lambda^n} \mathrm{e}^{-\beta U((x)_p,(y)_n)}\,\mathrm{d}(y)_n}{\Xi_\Lambda(z)} \right\}_{p=1}^{L}. \tag{54}$$

Let us consider the one point correlation functions $\hat{\rho}_\Lambda(x_1)$

$$\hat{\rho}_\Lambda(x_1) = \frac{\sum_{n=0}^{L-1} \frac{z^{n+1}}{n!} \int_{\Lambda^n} \mathrm{e}^{-\beta U((x_1)(y)_n)}\,\mathrm{d}(y)_n}{\Xi_\Lambda(z)}. \tag{55}$$

We integrate both sides of (55),

$$\int_\Lambda \hat{\rho}_\Lambda(x_1)\,\mathrm{d}x_1 = \frac{\sum_{n=0}^{L-1} \frac{z^{n+1}}{n!} \int_{\Lambda^{n+1}} \mathrm{e}^{-\beta U((x_1)(y)_n)}\,\mathrm{d}(y)_n\,\mathrm{d}x_1}{\Xi_\Lambda(z)}, \tag{56}$$

and find the following estimation

$$\int_\Lambda \hat{\rho}_\Lambda(x_1)\,\mathrm{d}x_1 \leqslant L\frac{\sum_{n=0}^{L-1} \frac{z^{n+1}}{n+1!} \int_{\Lambda^{n+1}} \mathrm{e}^{-\beta U((x_1)(y)_n)}\,\mathrm{d}(y)_n\,\mathrm{d}x_1}{\Xi_\Lambda(z)} \leqslant L. \tag{57}$$

In the thermodynamical limit we obtain

$$\frac{1}{V(\Lambda)}\lim_{\Lambda\to\infty}\int_\Lambda \hat{\rho}_\Lambda(x_1) \leqslant \lim_{\Lambda\to\infty}\frac{L}{V(\Lambda)} = \rho_{\max} \tag{58}$$

where $\rho_{\max}$ denotes the maximal density

$$\rho_{\max} = \lim_{\Lambda\to\infty}\frac{L}{V(\Lambda)} \tag{59}$$

and L is the greatest number of spheres with radius r placed in the domain Λ, and $V(\Lambda)$ is a volume of the domain Λ. r was chosen to keep the volume equal to one in every dimension. Since the correlation functions $\hat{\rho}$ in the thermodynamical limit does not depend on x_1, we get

$$\hat{\rho}_1 \leqslant \rho_{\max} \leqslant 1. \tag{60}$$

In the case of the n-point correlation function we use the following inequalities proved by Dobrushin and Minlos [9] for the hard core potentials:

$$\int_{\Lambda^k} \mathrm{e}^{-\beta U((x)_p,(y)_k)} d(y)_k \geqslant d' V(\Lambda) \int_{\Lambda^{k-1}} \mathrm{e}^{-\beta U((x)_p,(y)_{k-1})}\,\mathrm{d}(y)_{k-1}. \tag{61}$$

So we obtain

$$\begin{aligned}\int_{\Lambda^k} \mathrm{e}^{-\beta U((x)_p,(y)_k)} d(y)_k &\geqslant d' V(\Lambda) \int_{\Lambda^{k-1}} \mathrm{e}^{-\beta U((x)_p,(y)_{k-1})}\,\mathrm{d}(y)_{k-1} \\ &\geqslant Ld \int_{\Lambda^{k-1}} \mathrm{e}^{-\beta U((x)_{p+1}(y)_{k-1})}\,\mathrm{d}(y)_{k-1}\end{aligned} \tag{62}$$

where $d = d'\rho_{\max}^{-1}$.

From the above inequalities we obtain the following estimation of the n-point correlation functions:

$$\frac{\sum_{k=0}^{L-p} \frac{z^{p+k}}{k!} \int_{\Lambda^k} \mathrm{e}^{-\beta U((x)_p,(y)_k)}\,\mathrm{d}(y)_k}{\Xi_\Lambda(z)} \geqslant d\frac{\sum_{k=1}^{L-p} \frac{z^{p+k}}{k!} \int_{\Lambda^k} \mathrm{e}^{-\beta U((x)_{p+1},(y)_{k-1})}\,\mathrm{d}(y)_{k-1}}{\Xi_\Lambda(z)}. \tag{63}$$

So in the thermodynamical limit we get

$$\hat{\rho}(x)_p \geqslant \mathrm{d}\hat{\rho}(x)_{p+1}. \tag{64}$$

Thus

$$\hat{\rho}_1 \geqslant d^p \hat{\rho}(x)_{p+1}.$$

From above we conclude that

$$\|\hat{\rho}\| \leqslant 1. \tag{65}$$

Therefore, we have proven the following

Theorem 1. *Let us assume that $\hat{\rho}$ fulfils the KS equation in the infinite volume for the non-negative potentials; then the operator*

$$(\lambda I - K)^{-1} = -(N + \lambda N^2 + \cdots) - \hat{\rho} \prod [(KN - I)(I + \lambda N + \cdots)] \tag{66}$$

is the resolvent of the KS for $z = \lambda^{-1} > \mathrm{e}^{C(\beta)d^{-1}}$.

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